# TEACHING TRIGONOMETRY

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## Two methods of introducing trigonometry

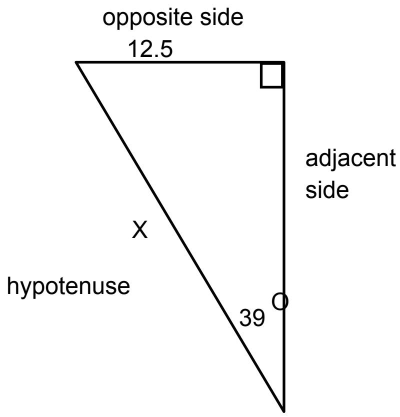
Since the advent of the "new mathematics", two methods of teaching introductory trigonometry have been used in Victorian schools. Originally, introductory trigonometry was taught by the ratio method, where trigonometric functions are defined as the ratios of pairs of sides in a right angled triangle. From the early 1960s, an alternative "modern" method was advocated by some educationalists (Trende, 1962; Willis, 1967) as a more desirable way for students to learn and understand the topic. This approach, known as the unit circle method, defined cosine and sine as the x and y co-ordinates of a point on a unit circle. Both methods are outlined in more detail below. Currently there are text books promoting each way of teaching trigonometry and a few try to blend the two approaches.

The question of which method is better was hotly debated in the pages of Vinculum and elsewhere in the 1960s and 1970s. Some people advocated each method and others favoured a combination of the two methods. Since 1982, the unit circle method has apparently been the preferred method for the Year 9 and 10 mathematics curriculum in Victoria (Secondary Mathematics Committee, 1982*; Geometry Everywhere,* 1991).

In this article, we discuss a research study which compared the two methods of teaching basic trigonometry. We wanted to see which promotes better understanding of the underlying concepts and mastery of skills. In the introductory phase of trigonometry which we are concerned with here, the main skills are to calculate the sides and angles of right angled triangles from known sides and angles.

### Ratio method.

The ratio method arises from observation, probably known to the Babylonians 3000 years ago, that the similarity properties of triangles can be used to find lengths and angles of triangles, and hence unknown lengths and angles in a variety of other figures. It is naturally allied with measurement and surveying. For the ratio method, the trigonometry functions are defined as the ratios of lengths of the sides in right angled triangles. For example, the sine of an angle is defined as the ratio of the length of the "opposite side" to the length of the hypotenuse. Students are often taught to remember the definitions of the ratios perhaps using a mnemonic such as SOHCAHTOA (<u>S</u>ine =<u>O</u>pposite + <u>H</u>ypotenuse etc). The named triangle sides are shown in Figure 1, along with one of the hardest calculations that students are expected to do at this early stage. Note that the difficulty of this example arises from the difficulty of solving the algebraic equation.




Find the length of the hypotenuse, x.

$$\sin 39° = \frac{\text{length of opposite}}{\text{length of hypotenuse}}$$

$$\sin 39° = \frac{1.25}{x}$$

$x = 12.5 + \sin 39°$

$x = 12.5 + 0.629$

$x = 19.873$

(OR more accurately, x = 19.863 if the value of sin 39° is not truncated before the final step)

*Figure 1. Naming the sides of a right-angled triangle and a hard calculation.*

**Unit circle method.**

When first introduced as part of the "new maths", the unit circle method emphasised the nature of the trigonometry functions as functions taking real numbers to real numbers. One of its virtues was seen to be that no reference to angles or triangles was required. Practical applications to meaurement were not the primary motivator although the solution of triangles was noted as "an interesting and useful outcome" (Dooley, 1968).

The trigonometry functions were defined as *functions* of a *real variable.* The angle A in Figure 2, for example, is defined as the length (a real number) along the circumference of a circle of radius 1, from (1,0) to P and the sine is the y-coordinate. This definition avoided the undefined notion of angle. One of the aims of new maths was to use mathematical language more precisely, so this was thought to be a desirable feature. Cosine and tangent are similary defined as lengths. Even though this is the definition which applies to all quadrants of the circle, current Victorian textbooks only use the first quadrant.

Whereas the ratio definitions arise naturally from applications to mensuration and surveying, the unit circle definitions lean more naturally to applications to periodic phenomena, such as simple harmonic motion. Both of these sources of applications are important. Our question here is not which definition should be used - we believe students as they learn more advanced mathematics need both. Our question is which method should be used to introduce trigonometry, generally in Years 9 and 10.. In answering this question, it is essential to consider what is the main goal of the teaching. Current textbooks and the *Curriculum and Standards Framework* see the main goal for introductory trigonometry as being the solution of right angled triangles.

How then is the unit circle method used to solve triangles? To find lengths and angles in right angled triangles, the given triangle is compared with the standard reference triangles (which students are to learn) and then properties of similar triangles are used for calculation. In order to avoid the difficulties of solving algebraic equations, especially with the unknown in the denominator, recent texts adopt a variation of the unit circle method which uses a scale factor comparing the given triangle with the reference triangle. Figure 3 illustrates the procedures involved in a case where it is difficult to visualise the scale factor.

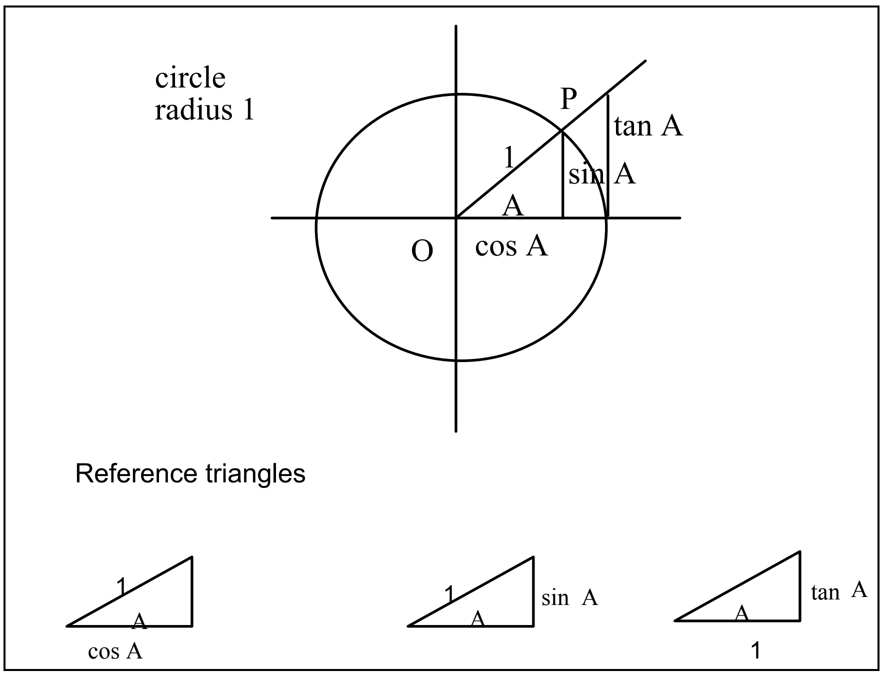


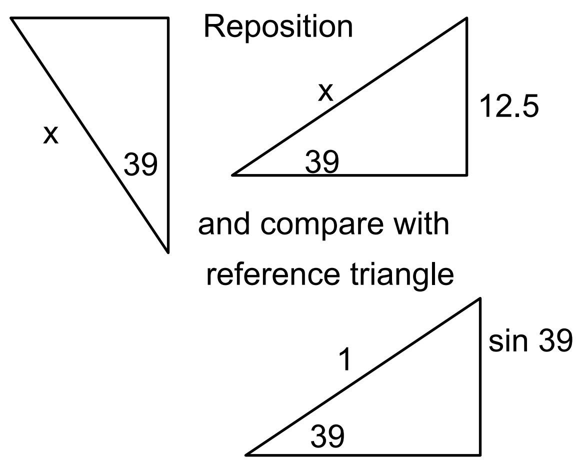

*Figure 2. Unit circle definitions of trigonometric functions based on the angle A and reference triangles.*

## The teaching experiment

We randomly allocated eight classes of Year 10 students at the school where the first author taught to two groups. Ninety students in four classes were taught basic trigonometry using the ratio method and the other four classes (88 students) were taught by the unit circle method. It happened that the two teachers who taught more than one year 10 class (Teacher 1 and Teacher 2 in Table 1 below) were both allocated one class of each teaching method. This enabled any potential teacher effect to be scrutinised. Based on a Year 10 common test done before the experiment, we found that overall the two teaching groups contained students of similar ability .

An entire teaching package for each method was prepared. The teachers were instructed about the methods and the teaching strategies, which included practical work and an outdoor investigation. The emphasis in both cases was on concept development, learning with understanding, enjoyment and skill development. As far as possible the lessons for the two methods were matched in teaching styles and content. The exercises which the students attempted were identical, but ordered differently to fit in with each teaching method. Twenty 45 minute lessons were devoted to teaching this unit of work.

Two tests were administered before and after the teaching. First, an arithmetic and algebra test was given to determine the level of proficiency of selected algebraic and arithmetic skills. The items identified students with division misconceptions and problems with algebra.




To find the length of the hypotenuse, x.

Reposition the triangle and match with reference triangle, as shown.

Similar triangles are used to match corresponding sides and set up scale factor

x =scale factor * corresponding side

In this case,
x = scale factor* 1
x= 19.863

*Figure 3. A difficult example using the unit circle method.*

The second test was on trigonometry. It contained 12 questions, each requiring the calculation of a nominated side length in a right angle triangle and some items asking students about their enjoyment of the unit and other attitudes.. About six weeks later, four similar questions were given as part of the end-of-year examination. Each trigonometry question on the test was awarded three marks, one for identification of the correct trigonometry function, one for the correct formulation and transposition of the equation to solve (written down or implied by further correct calculation) and one for performing the calculation correctly. If the student initially selected the wrong trigonometry ratio he/she could still gain two marks. Each student therefore received a score out of 36. The students had studied a short unit on trigonometry twelve months previously, using the ratio method. It was very disappointing that on the trigonometry pre-test, which was given without warning to the 178 students, all but four students scored zero.

## Results

*Overall performance on the trigonometry test.* Trigonometry performance was defined as the improvement from the pre-test to the post-test. The maximum score was 36. Individual class scores are given in Table 1. The students taught by the ratio method performed significantly better than the unit circle group. A second measure of success was the higher degree of retention demonstrated by the students on the trigonometry questions on the end-of-year examination. Ratio students obtained a higher mean score, again statistically significantly higher than the unit circle students. Within each teaching method group, Table 1 shows that the class means show little variation. Factors such as different class compositions and teachers have not influenced trigonometry performance as much as the method of teaching. Further analysis showed that the mean scores of students in every ability group (as measured on a common Year 10 test given earlier) was higher for the ratio method than for the unit circle method. Indeed, the mean of the lowest ability group in the ratio method was higher than the highest mean for the unit circle ability groups.

*Choosing the correct trigonometry function.* Students taught by the ratio method were more capable of identifying the correct trigonometry functions. The ratio students identified an average of 11.08 of the 12 functions correctly, with a median score of 12. In contrast, a large group of unit circle students identified very few functions. There are several key skills involved, but the rtatio method is clearly simpler. For the ratio method, the student needs to identify the sides in the triangle and then select the correct function. Most students reported using the mnemonic SOHCAHTOA to help them do this. In contrast, the unit circle method requires a complex inter-related set of procedures. The student must re-orient the given triangle if necessary, recall the unit circle diagram and identify which reference triangle matches the re-oriented triangle. Approximately one third of students could not correctly re-orient the triangle and nearly half were unable to draw correct reference triangles. The reference triangle for tan was the most difficult.

Table 1.
*Class means (and standard deviations) for trigonometry performance*

| Ratio Method Score out of 36 | | Unit Circle Method Score out of 36 | |
|---|---|---|---|
| Class | Mean | Class | Mean |
| Teacher 1 | 31.35 | Teacher 1 | 22.60 |
| Teacher 2 | 29.13 | Teacher 2 | 19.26 |
| Teacher 3 | 32.32 | — | |
| Teacher 4 | 32.76 | — | |
| — | | Teacher 5 | 21.82 |
| — | | Teacher 6 | 20.48 |

*Ability to formulate and transpose the equation.* The equation was considered to be correctly formulated if it was actually written down or it was implied by the student obtaining the correct answer for the selected ratio. For the ratio method, a correctly transposed equation was expected (i.e. in the form x = . . .) and for the unit circle method the scale factor needed to be determined together with the equation to calculate the side length of the triangle. Questions involving only multiplication were simple to formulate (e.g. find the length of a side opposite a given angle of 20° if the hypotenuse is known). These were completed successfully by 94% of ratio students and 68% of unit circle students. However questions where division was involved (e.g. finding the hypotenuse from the sine of an angle as in Figures 1 and 3) the formulation and transposition was more difficult and the success rate was 77% for ratio students and 38% for the unit circle students. The most common mistake for both methods was to multiply instead of divide. Ratio students were more successful with the division questions than unit circle students were with the multiplication questions. Students of low ability were particularly more successful solving the problems involving division using the ratio method. These results clearly demonstrate that the ratio method of teaching provides all students and in particular the lower ability students with an effective way to overcome their very real difficulties associated with the solution of more complex equations. Calculation mistakes were not method dependent and are a minor factor in trigonometry errors.

*Improvement in attitude.* At the same time as the trigonometry tests were given, students were asked to rate their liking for trigonometry on a five point scale. The ratio students had a singnificantly greater mean improvement in attitude than the unit circle students, again at every ability level. Students liked success.

*Improvement in solving algebraic equations.* The scale factor version of the unit circle method is promoted by teachers and texts because it avoids solving equations where the unknown is in the denominator and formal equation solving methods are demanded. On the other hand, the ratio method requires students to solve such equations. Eight items on solving equations with decimals were on the arithmetic and algebra pre- and post- test. Examples are $0.025x = 5$, $30 = 0.003/x$ and $25 = x /0.004$. As expected because of the practice involved, the ratio students showed significantly more improvement than the unit circle students. The greatest improvement was seen with the low ability ratio students.

## Summary

This study has provided substantial evidence tthat the ratio method students were better able to master the skills required and, with the greater success, came a greater improvement in attitude to the subject. The advantage the ratio students gained lasted through to the end-of year examination. The study showed that the teaching method, not the teacher, was the dominant effect. Students of all ability levels performed better with the ratio method, but the low ability students benefited the most.

Two procedures were critical to success on the test items. The first is an ability to identify which trigonometry function is appropriate for the problem. To identify the correct function to select many teachers, including those in this study, use a very simple, easily remembered mnemonic SOHCAHTOA. In contrast for the unit circle method there is a set of procedures required for re-positioning the triangle and matching it with a reference triangle. Not surprisingly, since the unit circle method is fraught with multiple opportunities for mistakes, there was a highly significant difference in students' ability to select the correct trigonometry function.

The second procedure involves formulating the correct trigonometric equation and transposing it so that it can be solved. For the ratio method, this is the step where most of the difficulties occur as the students may need to solve complex equations. Proponents of the unit circle method have sought to avoid this by adopting the scale factor method but this sseems to be harder than transposing the equations as required by the ratio method. Although formulations involving division were difficult for students in both methods, ratio students were more successful. Some transfer of learning, an improvement in the ability of ratio students to solve equations with decimals, was also noted.

### So what should a school teach?

It is absolutely clear that if a school sees the principal application of trigonomety ideas in Years 9 and 10 to be to the solution of triangles and the associated practial applications then the ratio method should be used. This is reasonably clear just from analysis of the steps to be performed in each method and completely clear from the results of this study, which were stronger than we had expected.

However, the items on the test we used were skill exercises of only one type, albeit on the main target of the teaching - solving triangles. Students were not tested for conceptual development, nor to see if foundations had been laid for later extension of the definitions of the trigonometric functions beyond the first quadrant. It is possible that the unit circle students may benefit in the long term in these aspects, but their generally very low facility in what was tested makes this seem very unlikely. The shockingly low retention of previous learning, as judged by the pre-test, also indicates caution before choosing any method on the basis of untested long-term benefits.

The unit circle definition of trigonometric ratios has many nice features. It is ideal for extension beyond the first quadrant, so it is particularly odd that none of the textbooks which use the unit circle definition go beyond the first quadrant. The unit circle provides concrete meanings for the trigonometric functions, as lengths of intervals which can be directly measured. The problem with the unit circle method is not the unit circle. Rather it is the complex procedure that are required to use the resulting definitions to solve triangles. In recommending the unit circle approach, curriculum developers opted for the more difficult path, with promise but no evidence of any long term benefit. Moreover, the scale factor version of this approach which is designed to eliminate one major difficulty has only replaced it by another of equal or greater magnitude. Since almost all of the curriculum goals that lead to the introduction of the unit circle method are now longer regarded as important at Years 9 and 10, the unit circle method has been left stranded high and dry when the curriculum tide has gone out.

Student learning is not a haphazard affair but is controlled by factors such as the teaching method together with a range of teacher and class influences. On balance, our preferred approach is that basic trigonometry be introduced using the unit circle trigonometric function definitions (and in more than one quadrant!), connecting them to the ratio definitions and then adopting the techniques of the ratio method for the solution of triangles. This type of approach is adopted by Ganderton, McLeod and Creely in *Mathematics For Australian Schools Year 10* (1991, pp. 258-259) and some other texts. This may enable students to experience real, concrete meanings for trigonometry definitions, lay some foundations for more advanced work and also have the thrill of successful performance in mathematics. It is clear that proposed curriculum change should be backed by empirical research, not only a philosophical debate and innovative teaching ideas.

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